Analytical readings' method as a tool to assess the word problem-solving process involving rational numbers

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Word problem solving in mathematics is essential, and the processes that students follow to solve them may vary. Consequently, teachers face the challenge of interpreting these strategies, while students sometimes have difficulties explaining their processes. The analytical readings' method is presented as a resource for the assessment of the word problem-solving processes, in addition to helping students develop skills in this area. In this context, the purpose of this study is to document reflection, especially among undergraduate mathematics students with an interest in teaching, on analytical readings as a tool to assess problem solving. Reflections highlight that the analytical readings' method allow assesses students' comprehension of the problem, their performed operations, the reasoning behind them, providing a comprehensive view of the thinking process.

Keywords: Word problems, analytical readings, word problem-solving process, assessment, rational numbers.

Introduction

Word problem-solving is considered a basic competence that promotes the development of arithmetic and algebraic thinking (Siegler et al., 2013). Rodríguez (2012) indicated that when solving a word problem, the student is allowed to "explore, experiment, analyse their progress, change course, reflect on what they have done, notice how they are thinking and approaching the task, etc." (p. 154).

Currently, word problems are an essential part of the mathematics education of every student around the world, and consequently, teachers must have resources to assess the solving process of these word problems. Therefore, our main objective is to promote the use of the analytical readings' method as a tool to assess the process of solving word problems with rational numbers, for this purpose the reflections of undergraduate mathematics students interested in teaching were documented.

It is clarified that it is not intended to promote the method as the best option or to say that it is the most efficient way of evaluation, but rather it is presented as another tool to evaluate, with advantages and disadvantages, which are documented in this study.

Theoretical framework

A word problem is a verbal description of a situation in which a question is posed and the answer to which can be found by applying mathematical procedures to the numerical data provided in it (Verschaffel et al., 2020). The word problem-solving process is understood as the student's mental activity from the moment he encounters a problem that must be solved until the task is finished;

considering ideas from Polya (1957), Puig and Cerdán (1988) proposed six phases in the resolution process: reading, comprehension, translation, calculation, solution, and verification.

The reading and comprehension phases were defined separately to emphasize the attention that should be placed on reading the problem at the beginning of word problem-solving instruction. However, reading and comprehension are not independent of each other, since they are aspects of the same operation that has the purpose of understanding the word problem.

The translation phase consists of the identification of the variables involved in the word problem, both known and unknown, and the relationship between them. In this way, three important aspects must be considered: what data is going to be handled, what operations or procedures will be carried out, and in what order.

The calculation phase refers to the execution of operations and algorithms. In this phase, the student's translation skills no longer intervene, but rather their algorithmic skills. It is important to note that Puig and Cerdán (1988) consider that "translation and algorithmic skills are usually independent of each other" (p. 14).

Finally, the solution phase consists of interpreting the numerical result obtained in terms of what is asked in the word problem, and in the verification phase we proceed to verify that said solution is adequate. The verification can range from something informal, such as seeing if the result makes sense (i.e. that there are no negative distances), to something more formal, such as substituting a value in an equation or solving the problem using a different procedure and compare the results.

One of variables associated with the word problem-solving process is the content variable. This is related to the mathematical meaning of the problem. The content variable in this study is related to rational number as operator. These numbers are associated with diverse uses. Based on the phenomenological analysis of fractions proposed by Freudenthal (1983), and the interpretation of Valenzuela (2018), five uses of fractions are distinguished at an abstract level: as fracturer, comparer, measurer, operator, and number. As said before, this work focuses on the operator aspect, that is, when the fraction acts on a quantity by expanding or reducing it. The operator aspect of the fraction is related to the phenomena of reproducing, reducing, enlarging, shrinking, expanding, contracting, etc. This operator aspect of the fractions can be extended to the diverse ways to express a rational number, such as decimal notation, percentage, and ratios.

Analytical readings' method

Taking as reference the analysis-synthesis method and the Cartesian method (Puig & Cerdán, 2014), the analytical readings' method to solve word problems is proposed, which is used indifferently for algebraic or arithmetic resolution processes. The method consists of the following five steps: reading the problem, forming a dictionary of quantities, building a tree graph, calculation, and solution-verification (see Figure 1).

In the first step, a first reading of the complete problem is done. Then it is read sentence by sentence, identifying the data presented and the verbs associated with them. Also, the main unknown quantity must be identified, that is, what needs to be calculated.

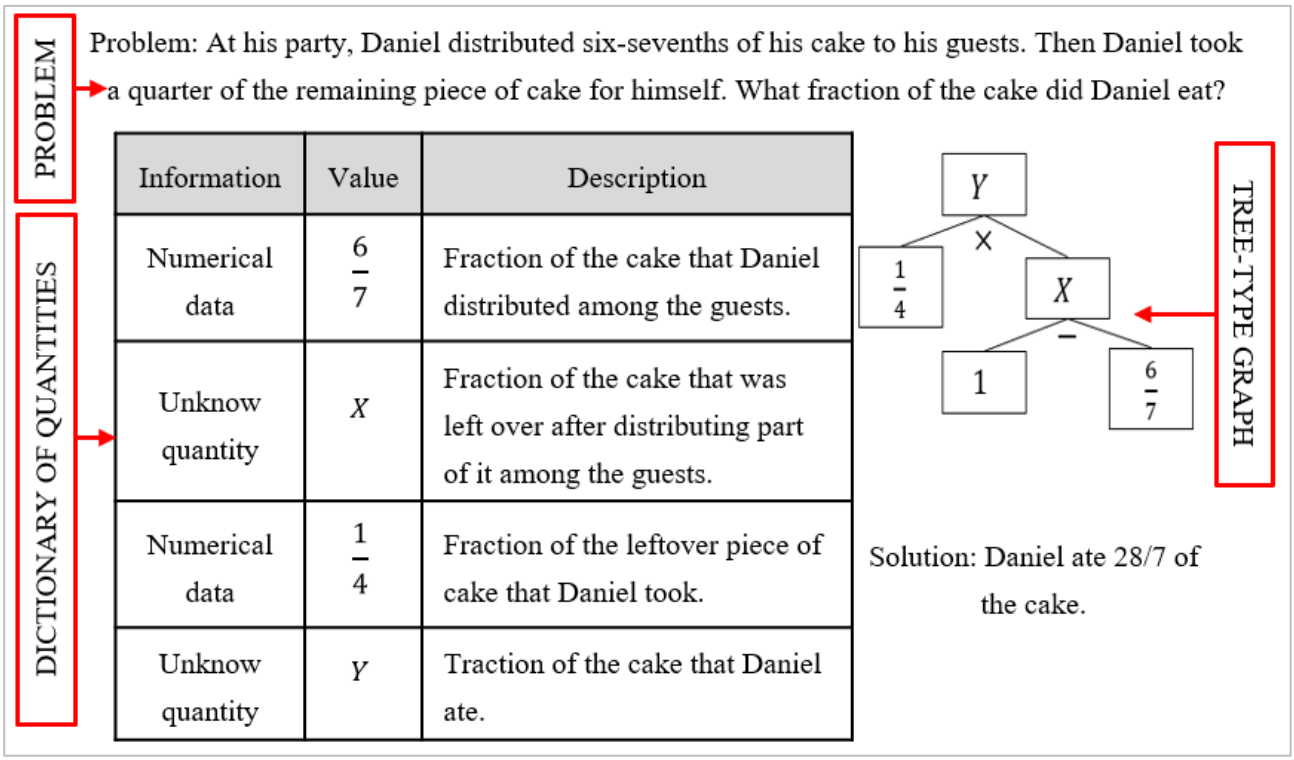

Figure 1: Example of the Analytical readings' method

To form the dictionary of quantities, the known and unknown variables that appear in the word problem are identified. This is organized in a table with three columns. The first column indicates whether the information is a given numerical data or an unknown value that must be calculated. In the second column, the numerical value of the data or what the unknown variable will be called is indicated. In the third column, a brief description of the variable is given.

The third step is to build the tree-type graph (Figure 2) following the rules of analysis-synthesis so that at the end all the numerical data in the problem and the main unknown quantity are obtained.

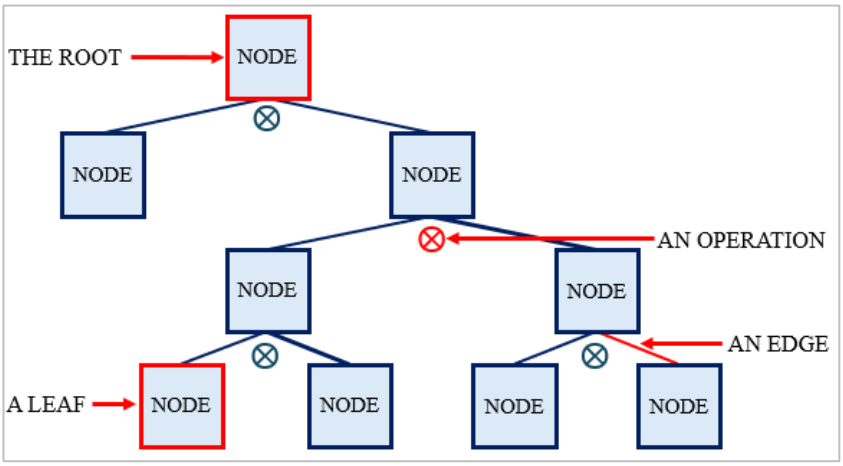

Figure 2: Tree-type graph

A tree-type graph has certain unique decomposition properties that allow obtaining a visual reproduction of the strategy followed in the resolution process (Roy & Roth, 2015). A tree is a data

structure consisting of nodes connected by edges, with the unique property of not containing cycles. For the analytical readings' method, we specifically use a rooted binary tree, in which there is a topmost node, commonly known as the root of the tree, and each node is linked to cero or two successor nodes. The nodes without successors nodes are called leaves.

In the fourth step of the method, the calculations that appear in the graph will be conducted, in the order that it shows. Finally, the fifth step is to write the result in terms of what is asked and verify it. In the Figure 1, an example of the steps followed in the analytical readings' method to solve a problem in shown. It should be noted that during the implementation of the analytical readings' method, the transit between its steps should not follow a strict linear order, it is always possible to go back and forth between the different stages.

Methodology

This qualitative research documents how mathematics students conceive the analytical readings' method as a tool to assess the word problem-solving process. In the study 13 students of the bachelor's degree in mathematics at the University of Guadalajara were considered. This is a group with affinity for teaching. The intervention consisted of three sessions that were part of an optional subject offered to the students. The worksheets, surveys, and essays were considered for the analysis. These were complemented with field notes, audio, and video recordings.

In the first session, the analytical readings' method was studied and explored through a document that simulated the solution of a word problem solved by a student -Paula-, reflecting in the process the dictionary of quantities, the tree-type graph, and its solution. For this exploration, teams of 3 or 4 members were formed. Teams are instructed to discuss what the student meant by what she illustrated, what her procedure was, and whether her answer is correct. Afterwards, their reflections were presented to the class. Next, the phases in the word problem-solving process proposed by Puig and Cerdán (1988) were explained. The steps that make up the analytical readings' method were delineated, while inviting the students to share their observations, contributions, and doubts. In this session, two word problems were given to the students to solve and practice. For each problem, a student goes to the board to solve it using the method learned, with comments and suggestions from the other students and the teacher's guidance.

In the second session, the students began by solving a problem in teams through the analytical readings' method as a review, comparing and discussing the results as a group. Subsequently, the different aspects of rational numbers and diverse forms to express them were exposed and discussed. Four problems were answered in teams using the diverse forms to express the rational numbers studied using the analytical readings' method (word problems involving fractions, ratios, percentages, and decimal notation). The dictionary of quantities, tree-type graph and solution of each team were compared and discussed.

In the third session, the relationship between the analytical readings' method and the word problem-solving process was discussed in a debate moderated by the professor. And finally, an anonym survey was carried out to identify students' opinions on the use of the method.

As a final assignment, the students were asked to write an essay answering the following questions: 1) How is the solver's thought process reflected when solving a word problem in each of the steps of the analytical readings' method? 2) How is each phase of the problem-solving process related to the steps of the analytical readings' method? 3) What is the usefulness of the analytical readings' method as an assessment tool for teachers?

Results

The results are divided into three sections: the relationship between the steps of the analytical readings' method and the phases of the word problem-solving process identified by the students, their opinions on the advantages and disadvantages of the method as an assessment tool, and their views on how this method adapts for word problems that involve each form to express the rationale numbers seen during the intervention.

Relationship between the steps of the analytical readings' method and the phases of the word problem-solving process

As described in the methodology, during the third session, the relationship between the analytical readings' method and the problem-solving process was discussed by the students. The relationships found by the students are summarized in Table 1.

Table 1: Relationship between the steps of the method and the phases of the problem-solving process

Steps of the analytical readings' method	Phases in the word problem-solving process that are related
Reading the problem	The reading and comprehension phases are related with this step of the analytical readings' method since the student is instructed to identify the question posed and the main unknown quantity, that is, what you should be calculated.
Forming a dictionary of quantities	The comprehension phase is also related with the step of forming a dictionary of quantities, because it helps to define the starting point (given numerical data) and know how to classify the information that you consider should be used to reach the solution. Also, a fundamental part of the translation phase is the identification of the variables involved in the word problem, which are reflected in the dictionary of quantities in an orderly manner.
Building a tree graph	Other aspects of the translation phase are what operations or procedures will be carried out and in what order? these are reflected in the tree graph.
Calculation	The step of calculation in the analytical readings' method, where the calculations that appear in the tree graph are conducted in the in the order that it shows, is equivalent to the phase of calculation in the problem-solving process, when operations and algorithms are executed.
Solution-verification	The solution-verification step of the method is analogous to the solution and verification phases of the word problem-solving process.

Advantages and disadvantages of the method as an assessment tool

During the second session, when the teams presented and compared their dictionary of quantities, tree-type graph and solution of four different problems, each one using the diverse forms to express the rational numbers, a group dynamic of assessment was carried out. Each team tried to describe the thought processes of their partners by reading the method. Subsequently, each student gave their opinion on the advantages and disadvantages of the analytical readings' method as an assess tool in their essays and anonymously in the survey. These opinions are summarized in the Table 2.

Table 2: Advantages and disadvantages of the analytical readings' method as an assessment tool

Advantages	Disadvantages
This method works as a powerful assessment tool, through which it is not only possible to assess mathematical knowledge but also to assess reasoning, logic and even reading comprehension skills. This tool can provide extra information about the students' thinking process. As teachers, the steps of the analytical readings' method help us identify the student's points of deficiency. The incorrect construction of the dictionary of quantities suggests problems with the reading and/or understanding of the text. The erroneous layout of the graph indicates problems with the translation of the text into mathematical processes and the incorrect calculation of the data reveals deficiencies in the student's arithmetic knowledge. In addition, the method can be very useful for teachers when grading, as it encourages students to express their procedure in an orderly manner.	The analytical readings' method can be laborious due to the details to be specified in each step; this mainly has two disadvantages: By virtue of its slow nature, the method is restricted to limited uses in a classroom environment where deadlines must be met. Before it can be used as an assessment tool, students must be taught to work with this method. When the method is introduced for the first time, the solution process can be confusing at first, especially the tree-type graphs, since it is a new way of representing operations. Therefore, extra sessions would be needed for the group to become familiar with the method.

An example, when discussing the procedure made by a team to solve the next problem:

> Problem 1. Melisa and Gerardo are going to paint the house where they both live. Melisa can paint the house in 3/4 of a day, while Gerardo is able to paint the house in 5/6 of a day. If they work together, what fraction of a day will it take them to paint the house?

The students highlighted the importance of seeing the variables in the dictionary of quantities. They argued that if the variable written in the tree graph (Figure 3) had been written in the dictionary, it would have allowed them to know how their classmates were interpreting the sum of the fractions of days that Gerardo and Melisa took to paint the house individually, why they carried out this operation, and therefore better understand what the error was in understanding the problem that led to the wrong answer. During the discussion, it was also mentioned that the tree graph allowed them to see how the fractions 3/4 and 5/6 were related, which led their classmates to an error.

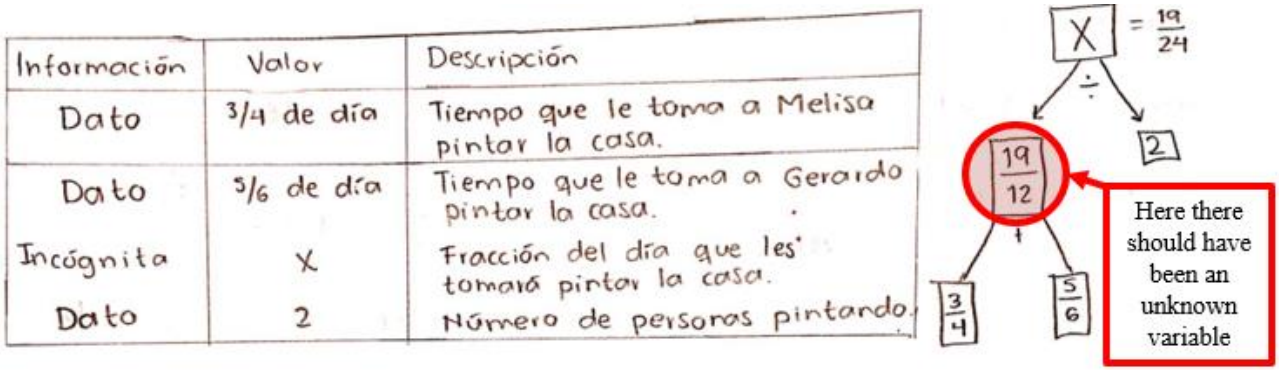

Figure 3: Procedure made by a team while solving problem 1

The use of the analytical readings' method for solving word problems that involve diverse forms to express the rational numbers.

As indicated previously, during the intervention the students worked with four different word problems, which involved fractions, ratios, percentages, and decimal notation. In the survey, participants were asked to indicate how well they think the method is adapted to solve problems involving each of these four ways of expressing rational numbers, the options offered were: 1) *It does not fit well* -the explanation of my procedure feels forced when I use this method-. 2) *It fits well* -I feel that the method helped me explain my procedure in a clear and structured way-. 3) *Indifferent* -I don't feel like the method helped or hindered me in my procedure. Each answer had to be justified.

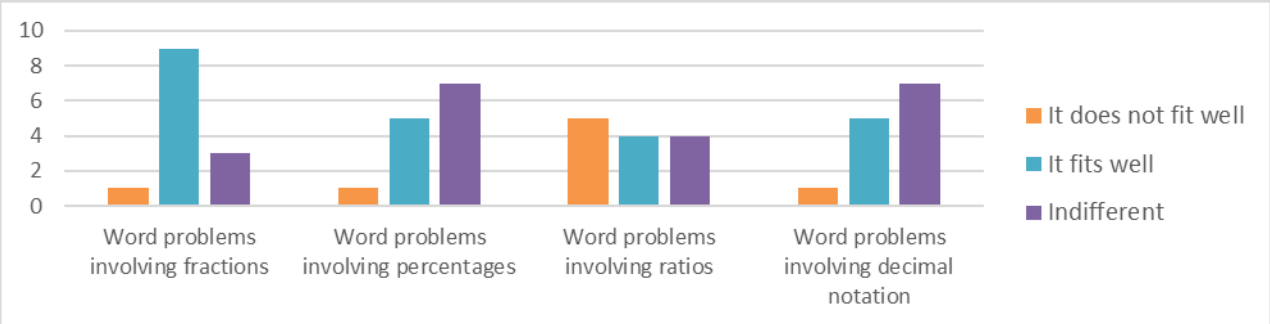

Figure 4: Students' opinion on the use of the method taking into account the content variable

It can be seen in Figure 4 that the mathematical content variable affects the students' perception of how well the method is adapted to capture their thinking process and word problem-solving process. In particular, students think that the analytical readings' method helped them explain their procedures in a clearer way when solving word problems that involve fractions, but this is not the case for those word problems that involve ratios.

Conclusions

The objective of this study was to test the analytical readings' method, regarding its use to assess the processes of word problems solving to document the reflections of a group of students. In this regard, there was evidence that the students considered the analytical readings' method as an assessment tool allows the student's mental activity and thought process to be clearly reflected in their procedure when solving word problems. As each step of the method is related to the phases in the word problem-solving process, it makes easier to follow the student's progress through each of the phases and, if

there is an error, determine at what point in the resolution process it occurred, and identify its possible causes. However, according to the students it must be considered that there may be some disadvantages, such as the difficult, and time it takes to use this method when solving a problem.

The use of the analytical readings' method discussed in this paper is as an assessment tool, but during the intervention a topic of discussion among the study participants was the potential of the method as a teaching tool. The advantages of learning to solve certain types of word problems, for example those involving fractions, were discussed. The possibility that this method offers was highlighted so that students can explain their procedures, as well as the development of skills to establish relationships between the quantities that appear in the problem. Although it was also mentioned that using this method for the first time could be a challenge for students.

References

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrech: D. Reidel.

Polya, G. (1957). How to solve it (2nd ed.). Princeton University Press.

Puig, L. & Cerdán, F. (1988). Problemas aritméticos escolares. Madrid: Síntesis.

Puig, L., & Cerdán, F. (2014). Acerca de carácter aritmético o algebraico de los problemas verbales. En B. Gómez & L. Puig (Eds.), *Resolver problemas: Estudios en memoria de Fernando Cerdán* (pp. 21-34). Valencia: Universitat de València.

Rodríguez, M. A. (2012). Resolución de Problemas. En M. D. Pochulu & M. A. Rodríguez (Eds.), *Educación Matemática: aportes a la formación docente desde distintos enfoques teóricos* (1ra ed., Vol. 1, pp. 153-174). Editorial Universitaria de Villa María, Universidad Nacional de Villa María; Universidad Nacional de General Sarmiento.

Roy, S., & Roth, D. (2015). Solving General Arithmetic Word Problems. En L. Màrquez, C. Callison-Burch, & J. Su (Eds.), *Proceedings of the 2015 Conference on Empirical Methods in Natural Language Processing* (pp. 1743–1752). Association for Computational Linguistics. doi: 10.18653/v1/D15-1202.

Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Sciences, 17*(1), 13-19. doi: 10.1016/j.tics.2012.11.004

Valenzuela, C. (2018). *Modelo de enseñanza para fracciones basado en la recta numérica y el uso de applets: estudio en comunidades marginadas* [Tesis doctoral]. CINVESTAV.

Verschaffel, L., Schukajlow, S., Star, J., & Van Dooren, W. (2020). Word problems in mathematics education: a survey. ZDM Mathematics Education, 52, 1–16. https://doi.org/10.1007/s11858-020-01130-4