Enhancing Teacher Training in Mathematics Education: A Model for a Semiotic Approach to Feedback and Interpretative Knowledge

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This research introduces a comprehensive model for a teacher training course centered on Semiotic Interpretative Knowledge (SIK) in mathematics education. Highlighting the critical need for specialized training, the course is designed to refine teachers' abilities to interpret student's responses through a semiotic lens, especially when conceptual knowledge remains hidden behind difficulties related to patterns of sign use. It focuses on equipping educators with advanced semiotic interpretation skills, thereby enhancing their capability to offer deeper, more meaningful, and effective feedback in mathematics classrooms. The model not only delineates the key features of the designed course but also lays a foundation for future investigations into its feasibility and impact.

Keywords: Teacher training, feedback, semiotic functions, Semiotic Interpretative Knowledge.

Research problem and rationale of the paper

In mathematics education research, the types of feedback that teachers spontaneously provide in their mathematics classrooms are investigated by several studies (e.g., Galleguillos & Ribeiro, 2019; Santos & Pinto, 2010; Stovner & Klette, 2022). The main aspects of feedback that emerge from such research concern conceptual, strategical, or procedural features, while the semiotic aspects related to sign use and production are never explicitly considered. As research shows (e.g., D'Amore & Fandiño Pinilla, 2007; Iori, 2018; Santi, 2011), interpreting student's reasoning requires a strong semiotic competence on patterns of sign use and production. Asenova et al. (2023a) define the notion of Semiotic Interpretative Knowledge (SIK) as "the knowledge needed by teachers in order to interpret students' answers (…), and to give appropriate feedback to them, when conceptual knowledge is hindered, and thus remains hidden behind difficulties related to patterns of sign use and production" (p. 11). In Asenova et al. (2023b) an investigation carried out with 180 Italian prospective primary school teachers shows that prospective teachers spontaneously use a wider web of semiotic resources when they are asked to provide feedback to students, rather than when they are asked to interpret the student's solutions for themselves. Prospective teachers seem to implicitly assume that feedback effectiveness grows with increasing use of semiotic resources, but the above-mentioned study shows also that they often fail in this because of a lack of awareness on the semiotic transformations involved in their feedback. The necessity to consider SIK as a kind of mathematical knowledge for teaching (Ball et al., 2008) and as an explicit content area in teacher training, specifically in reference to feedback effectiveness, goes with a lack of research on how to implement teacher training focused on SIK. This paper presents a model for teacher training courses which address this issue.

Theoretical framework

The semiotic dimension of mathematical knowledge for teaching

Starting from research related to the conceptualization of Mathematical Knowledge for Teaching (MKT) (Ball et al., 2008), Ribeiro and co-authors introduce the notion of Interpretative Knowledge (IK) as the part of the mathematical knowledge "that allows teachers to give sense to pupils' non-standard answers (i.e., adequate answers that differ from the ones teachers would give or expect) or to answers containing errors" (Ribeiro et al., 2016, p. 9).

The semiotic aspects of IK are still little explored, but at the same time research shows that a strong semiotic competence is indispensable for a cognitively meaningful mathematical activity. According to Duval (2017), in mathematics, ostensive references are impossible, as we cannot directly access mathematical objects through our senses. Conceptualization itself, called noesis, is identified in mathematics with a complex coordination of several semiotic systems (called semiosis), rooted in semiotic transformations within the same semiotic system (treatments) and semiotic transformations between different semiotic systems (conversions) (Duval, 2017). A semiotic system (or register) is defined by Duval (1995) and Ernest (2006) as composed by: (1) a set of basic signs that only have meaning when set against or in relation to other basic signs (e.g., the meaning of the digits within the decimal number system); (2) a set of rules for the production of signs, starting from basic signs, and for their transformation. According to Duval, D'Amore (2003) identifies conceptualization with the following semiotic functions, specific to mathematics: (1) choice of the distinctive features of a mathematical object and its representation in a semiotic system; (2) treatment in the same semiotic system; (3) conversion between semiotic systems. The management of such semiotic complexity, within the structure of semiotic systems and the processing of semiotic functions, comes up against Duval's famous cognitive paradox (Duval, 2017): on the one hand, the student comes to know the abstract mathematical objects only through the semiotic activity; on the other hand, such a semiotic activity requires the student's conceptual knowledge of the mathematical objects involved in it. According to Duval, a mathematical object, intended as an epistemic object, arises as the invariant behind treatments and conversions and thus requires the interplay of at least two semiotic registers.

For the reasons displayed above, taking into account the intrinsically semiotic nature of mathematical thinking, Asenova et al. (2023a) introduce the theoretical construct of Semiotic Interpretative Knowledge (SIK) as "the knowledge needed by teachers in order to interpret students' answers (be they standard or non-standard), as well as students' behavior, and to give an appropriate feedback to them, when conceptual knowledge is hindered, and thus remains hidden behind difficulties related to patterns of sign use and production, including individual creativity in sign use" (p. 11). SIK is a kind of MKT that is both subject- and pedagogy-related because the control of the semiotic functions is intertwined both with mathematical knowledge (noesis and semiosis are overlapped) and their implementation in the teaching-learning activity driven by the teacher.

The feedback dimension

Feedback is defined by Hattie and Timperley (2007) as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (p. 81). Galleguillos and Ribeiro (2019) investigate prospective teachers' ability to use IK in giving feedback: teachers were asked to solve a task in small groups and then provide feedback to chosen solutions given by students to the same task. These authors classify the provided feedback into four categories: (a) Feedback on how to solve the problem; (b) Confusing feedback: When the feedback seems to be correct, but it can be confusing for the student; (c) Counterexample as feedback; (d) Superficial feedback: The content of such feedback was insufficient (too broad or inconsistent) to allow the solver to understand its meaning. In Asenova et al. (2023b) the authors develop the kinds of feedback introduced by Galleguillos and Ribeiro, consistently with the notion of SIK. In particular, starting from answers given by prospective teacher to similar tasks as the ones proposed by Galleguillos and Ribeiro (2019), the authors categorized the collected feedback according to the implementation of the semiotic functions: type (i) - no mention of semiotic functions, which is framed by Galleguillos and Ribeiro's categories; type (ii) - use of semiotic representations confined to the recognition of the distinctive features; type (iii) - use of distinctive features and treatments; type (iv) - use of distinctive features, treatments, and conversions. The semiotic categorization of feedback does not provide levels of effectiveness per se, but it represents a tool able to tune sign use in producing and evaluating feedback, by identifying levels of complexity of semiotic activity. In this sense, it reduces the gap "between what is understood and what is aimed to be understood" (Hattie & Timperley, 2007, p. 82) on feedback in teacher training and it allows to consider SIK within the categories of mathematical knowledge for teaching.

Research questions and aim

A strong SIK is needed by teachers both to interpret student's solutions, especially when conceptual aspects are hindered by difficulties related to patterns of sign use and production, as well as to provide effective feedback (Asenova et al., 2023a, 2023b). Starting from this assumption and building upon the work of Ribeiro et al. (2016), the present study aims to present a design for a teacher training course on SIK. More specifically, the research question addressed here is: *What are the characteristics of a teacher training course which develops SIK in relation to feedback exchange and production?*

The characteristics of the course will be described, emphasizing their link to the theoretical framework introduced. Additionally, it will be discussed how these characteristics correlate with expectations regarding the processes engaged by the participants throughout the course. The development of a strong SIK on the part of the prospective or in-service teachers is a significant step towards enabling them to understand and be effective in the management and support of student's learning and strategies, particularly in relation to the use of semiotic functions. This proposal serves as a theoretical and methodological basis for future research on the feasibility and effectiveness of the training course described.

The model for SIK operationalization in teacher training courses

The basic structure of the proposed model is composed of five phases, each associated to an explicit goal (Table 1).

Table 1: **Model of a cycle of SIK operationalization in feedback in teacher training courses**

	Course phases		Main goal
1		Introduction to semiotic functions and their noetic correspondences (step 1) and to semiotic transformations (step 2)	Introduction of tools for the development of SIK
2		Task 1: Giving written feedback to student's solutions working in small groups	First implementation of SIK
3		Task 2: Written peer-to-peer evaluation to the feedback provided by another group according to criteria related to semiotc functions	Metareflection on SIK implementation in relation to semiotic functions
4		Exchange of feedback and reflection in the groups	Metareflection on SIK evaluation: analysis on one's own work
5		Feedback of the teacher educator during whole class discussion	Institutionalization

In the following the five phases are analyzed into detail and the relationships between the proposed tasks and expectations about the course are unraveled.

Phase 1. During the first phase the participants are introduced to the aspects related to the semiotic functions and their role in the recognition of the mathematical object as invariant behind treatments and conversions. In this phase the definition of conceptualization given by D'Amore (2003) referring to the semiotic functions is strongly used and exemplified. During step 1, focusing on D'Amore's definition of conceptualization, some examples are discussed with the participants recalling their attention to the properties of mathematical objects and the representations of their distinctive features. In Figure 1 four such examples are presented. In the first example (a), choosing the representation on the left highlights the distinctive feature of a fraction as a ratio between equal areas; choosing the representation on the right highlights the distinctive feature of a fraction as a ratio of areas of congruent figures. In the second example (b), only one of the representations ($\frac{2}{3}$ of $\overline{AB}$) highlights the distinctive features of a fraction as an operator (on a magnitude). In the third example (c), choosing the representation on the left brings out the distinctive features of the parallelogram as a quadrilateral with two pairs of parallel sides and, consequently, two pairs of congruent angles; choosing the representation on the right brings out the distinctive feature of the parallelogram as a quadrilateral, but by looking at it as an icon rather than recalling its properties as a bidimensional geometric figure. In the fourth example (d), the initial representation ($\frac{6}{12}$) emphasizes the distinctive features of probability as the ratio of favorable outcomes to total outcomes; the subsequent representation ($\frac{1}{2}$) still portrays probability as a ratio, but does so in a more abstract way: for each of the favorable outcomes, there are two possible outcomes; the third representation (50%) still carries some distinctive feature of probability as a ratio, but only mediated by the meaning of percentage ('per cent' as 'per hundred', from the Latin word 'cento', 'hundred'): there are 50 favorable cases out of 100 possible cases; the fourth representation (0.5) highlights the distinctive feature of probability as a real number in the interval [0;1]. This introduction is especially significant in raising teachers' awareness about the critical attention needed when selecting one representation over another, and the unique characteristics that such representations can either emphasize or conceal.

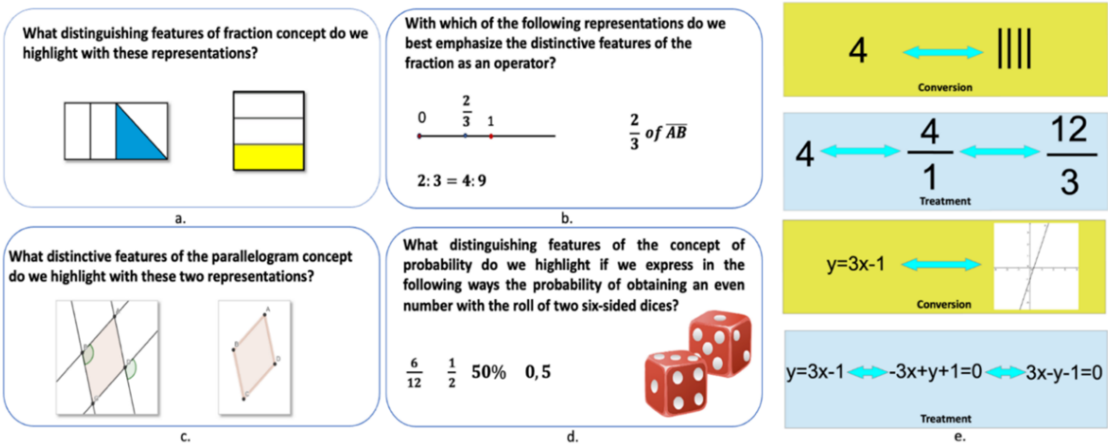

Figure 1: Examples of choice and representation of distinctive features of mathematical objects and of semiotic transformations to be discussed with the participants

Step 2 of phase 1 focuses on semiotic transformation and the participants are introduced to the meaning of treatments and conversions. For this purpose, examples such as those displayed in Figure 1e are presented and discussed with the participants. A conversion between 4 in the decimal arithmetic semiotic system and an iconic representation belonging to the semiotic system of pictograms is accomplished. Then, two treatments within the decimal arithmetic semiotic system are accomplished; in this system there are established rules (how to transform 4 in $\frac{4}{1}$ and $\frac{4}{1}$ in $\frac{12}{3}$) that allow to perform the transformations within the system. In the third row, a conversion between the algebraic semiotic system and the cartesian semiotic system is represented. It allows to evidence the correspondences between the distinctive features of the mathematical object in the semiotic systems: the coefficient '3' corresponds to the slope in the cartesian system; the constant '-1' corresponds to the intercept on the y-axis; each ordered pair $(x; y)$ of numbers that satisfies the equation $y = 3x - 1$ corresponds to a point on the line in the cartesian representation. It is important to point out these aspects because, according to Duval (2017), to support conceptualization, it is not enough to provide representations in different semiotic registers, but it is necessary to make explicit the correspondences between the distinctive features in the different registers. Furthermore, Duval (2017) stresses that conversions are necessary for the conceptualization of the mathematical object, but that the change of semiotic system often makes lose the meaning behind the performed transformations. Teachers often overlook this crucial aspect: they are already familiar with the concept that remains constant across various representations in different systems. However, students frequently miss this invariant and perceive the representations as distinct entities (e.g., students do not recognize a representation of the same object 'line' behind the algebraic and the cartesian representations). But not only conversions lead to a loss of meaning; as D'Amore and Fandiño Pinilla (2007) show, also treatments often lead to a loss of the invariant behind the transformations. Furthermore, it is important to stress that the choice of the distinctive features (properties of the object to be represented) is not independent on the choice of the representation system, as the possibility to represent strictly depends on the semiotic resources provided by the system (D'Amore & Fandiño Pinilla, 2007).

In summary, it is crucial for educators to offer representations in various semiotic systems for two reasons: firstly, identifying an invariant (mathematical object) across different contexts (semiotic systems) requires at least two such systems; and secondly, a single semiotic system is often insufficient to depict all the key features of a mathematical object that students need to understand to fully grasp the concept. The subsequent stages of the model are designed to assist participants in reflecting upon the semiotic perspective introduced within the context of exchanging feedback.

Phase 2. During the second phase of the model, the participants are asked to work in small groups on Task 1 that requires to provide written feedback to student's solution of a task. No specific guidelines are provided on the criteria to be adopted when giving feedback, allowing participants the freedom to use the methods they deem most suitable. It is important to choose tasks that drive the use of semiotic functions, particularly conversions involving symbolic language, natural language, and figural representations. Usually, solutions that contain errors are selected as they more effectively motivate teachers to offer feedback. However, correct yet unconventional solutions that encourage the application of semiotic functions are also suitable. For this purpose, the example proposed by Ribeiro et al. (2016) is particularly suitable (Figure 2), because it involves a nonstandard procedure and the use of only an iconographic register not involving symbolic or verbal ones. Similar problems and their solutions provided by students can be given, involving different registers and strategies.

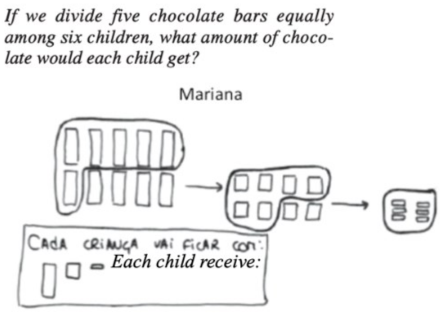

Figure 2: Mariana's solution to the task presented in Ribeiro et al. (2016)

Phase 3. In the third phase of the model, the groups are asked to exchange their feedback with another group and to work on Task 2. This requires them to give a peer-to-peer assessment to the feedback provided by the other group in Task 1. In this case, criteria for the evaluation-feedback are provided. These criteria are the following: (1) Does the choice of distinctive features to represent in the feedback seem appropriate? (2) Does the feedback involve treatments, i.e. semiotic transformations within the same semiotic system? (3) Does the feedback use conversions, i.e. transformations between different semiotic systems? (4) Are the representations chosen in the feedback related to the representation used by the student in the solution? Some examples are provided of how each criterium could be concretely exemplified in relation to the solution provided by the student: for instance, if the teachers want to support the use of a symbolic register in relation to fraction, feedback to Mariana's strategy must go toward this direction. Criteria are provided at this stage to directly foster metareflection on using semiotic functions. Additionally, evaluating others' feedback is believed to prompt a detachment from the content, resulting in a more genuine reflection (Grion & Tino, 2018).

Phase 4. In the fourth phase of the model, the groups return their peer-to-peer-feedback to the other group and reflect on the evaluation received from their peers. Providing feedback to another group fosters reflection and offers an opportunity to gain a fresh perspective on the task, enabling to revise and enhance one's own original solving strategies. This is mainly accomplished by reflecting on the evaluation of the appropriateness of the semiotic resources used in providing feedback to the student's solution as well as on their functionality with regard to supporting the development of the semiotic functions. In this sense, the feedback provided by others helps to bridge the gap "between what is understood and what is aimed to be understood" (Hattie & Timperley, 2007, p. 82) regarding the appropriateness of one's SIK.

Phase 5. In the fifth and final phase, feedback is provided to the participants by the teacher educator. This occurs in two distinct manners: firstly, through addressing questions that emerge during the activities of the previous phase, and secondly, through the discussion of general issues related to providing suitable feedback to students. Although it is recognized that there is no universally 'correct' feedback and 'ideal solutions' are intentionally not offered, an evaluation on the appropriateness of the implemented semiotic functions is carried out. The objective of this phase is the institutionalization of the new knowledge that has been developed. The focus should here be posed particularly on a comparison of the effectiveness of the chosen semiotic systems and semiotic representations, as well as on the semiotic transformations carried out in providing the feedback and their role in improving one's SIK.

The five phases of the basic model of SIK operationalization in relation to feedback have been presented. The model is conceived as cyclic: the repetition of the model on different problems allows work on semiotic functions in relation to various mathematical objects and varied strategies.

Conclusive remarks

A cohesive model aligned with the theoretical foundations of semiotic functions and IK has been crafted: the characteristics of the course have been presented allowing to work in the direction of defining structured design principles. The course design supports prospective and in-service teachers in consciously referring to semiotic functions while interpreting and providing feedback to students. This approach promotes openness to unconventional or incorrect strategies, highlighting effective conceptual or strategic elements that may be obscured by atypical or incorrect representation usage. The course structure demands focused communicative effort, especially regarding feedback mechanisms and representation choices. It underscores the importance of developing the ability and willingness to understand others' mathematical viewpoints. This comprehension is linked to feedback and interpretations within the SIK framework, recognizing that the selection of a semiotic register reflects an individual's conceptual categories for interpreting reality.

Overall, this contribution lays the groundwork for defining a set of design principles and assessing the effectiveness of courses aimed at developing SIK. It defines the crucial aspects in the foundation of the knowledge needed by the teacher for its development and thus the indicators for its assessment.

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